7. a)

$$\frac{\sqrt{a}-2}{a-4} \times \frac{\sqrt{a}+2}{\sqrt{a}+2} = \frac{a-4}{(a-4)(\sqrt{a}+2)} = \frac{1}{\sqrt{a}+2}$$

b) $$\frac{\sqrt{x+4}-2}{x} \times \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \frac{x+4-4}{x(\sqrt{x+4}+2)} = \frac{1}{\sqrt{x+4}+2}$$

c) $$\frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{x+h-x}{h\sqrt{x+h}+h\sqrt{x}}$$

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

1.2 The Slope of a Tangent

- The slope of the tangent to a curve at a point P is the limit of the slopes of the secants PQ as Q moves closer to P.

$$m_{tangent} = \lim_{Q \to P} (\text{slope of secant } PQ)$$

- The slope of the tangent to the graph of $y=f(x)$ at $P(a, f(a))$ is given by

$$m_{tangent} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

- To find slope of the tangent at a point $P(a, f(a))$
  - find the value of $f(a)$
  - find the value of $f(a+h)$
  - evaluate $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ ← difference of quotient

- we want h to be 0 at the end.

Ex. The slope of the tangent to the graph of $y=f(x)$ at point $(a, f(a))$ is $m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

when $h \to 0$
- $6+h = 6$
- but $\frac{6+h}{h} \neq 6$

- when h is 0, it is undefined
- when $h \to 0$, then a value can be defined

- we write $\lim_{h \to 0}$ to say limit as 'h' approaches 0.

ex. Determine the slope of the tangent if $f(x) = \frac{2x+1}{x}$ at the point $(1, 3)$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$f(1+h) = \frac{2+2h+1}{1+h}$ $\quad f(1) = 3$

$$= \lim_{h \to 0} \frac{\frac{2+2h+1}{1+h} - \frac{3(1+h)}{1+h}}{h}$$

← common denominator or rationalization

$$= \lim_{h \to 0} \frac{\frac{2+2h+1-3-3h}{1+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{1+h} \div \frac{h}{1}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{1+h} \times \frac{1}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{-1}{1+h}$$

← We can't replace 'h' with 0 because it is in the denominator so $m = \lim_{h \to 0} \frac{-1}{1+h}$ and $m = -1$

$$= \frac{-1}{1}$$

$$= -1$$

ex. $f(x) = x^2$ when $(5, 25)$

$$m = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

$f(5+h) = 5^2 + 5h + 5h + h^2$
$= h^2 + 10h + 25$
$f(5) = 25$

$$= \lim_{h \to 0} \frac{h^2 + 10h \cancel{+25} \cancel{-25}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(h+10)}{\cancel{h}}$$

- $a^2 - b^2 = (a-b)(a+b)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$$= \lim_{h \to 0} h + 10$$

$$= 10$$

ex. $f(x) = \sqrt{x}$ when $(9, 3)$

$$m = \lim_{h \to 0} \frac{f(9+h) - f(9)}{h}$$

$f(9+h) = \sqrt{9+h}$
$f(9) = 3$

$$m = \lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h}$$

$$m = \lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} \times \frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3}$$

$$m = \lim_{h \to 0} \frac{(9+h) - 9}{h(\sqrt{9+h} + 3)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{9+h} + 3)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{9+h} + 3}$$

$$= \frac{1}{\sqrt{9} + 3}$$

$$= \frac{1}{6}$$

ex. $f(x) = (x+2)^3$ $\qquad x = 1$

$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(3+h)^3 - 3^3}{h}$$

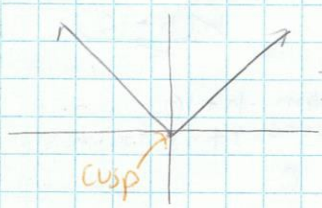
$$= \lim_{h \to 0} \frac{(3+h-3)((3+h)^2 + (3)(3+h) + (3)^2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{(h)}(9 + 6h + h^2 + 9 + 3h + 9)}{\cancel{h}}$$

$$= \lim_{h \to 0} (h^2 + 9h + 27)$$

$$= 27$$

ex. $f(x) = |x|$ $\qquad x = 0$



slope is undefined
- at cusp
- vertical tangent
- " asymptote
- hole (sometimes)

$$m = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - 0}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$\lim_{h \to 0^+} \frac{|h|}{h}$ $\quad\Big|\quad$ $\lim_{h \to 0^-} \frac{|h|}{h}$

$= \lim_{h \to 0} \frac{h}{h} = 1$ $\quad\Big|\quad$ $= \lim_{h \to 0^-} \frac{-h}{h} = -1$

$m+ \quad 0 \leq x \leq 0.01$

$m- \quad -0.01 \leq x \leq 0$

• if results are mixed, no limit will work so it is undefined

• m = at cusp undefined

↳ +, − → mixed result

∴ slope is undefined

$f(x) = x^2 + 2x + 5$

$m = \lim_{h\to 0} \frac{(a+h)^2 + 2(a+h) + 5 - (a^2+2a+5)}{h}$

$m = \lim_{h\to 0} \frac{h^2 + 2ha + \cancel{a^2} + \cancel{2a} + 2h + \cancel{5} - \cancel{a^2} - \cancel{2a} - \cancel{5}}{h}$

$m = \lim_{h\to 0} \frac{\cancel{h}(h+2a+2)}{\cancel{h}}$

$m = \lim_{h\to 0} 2a+2 \qquad m = 6$

$6 = 2a+2$

$\frac{4}{2} = 2a$

$a = 2$

$(2, 13)$

2.

$\frac{\frac{1}{3}(a+h)^3 - \frac{1}{3}a^3}{h} = \frac{\frac{1}{3}((a+h)^3 - a^3)}{h}$

$= \frac{\frac{1}{3}(\cancel{a+h-a})((a+h)^2 + a(a+h) + a^2)}{\cancel{h}}$

$= \frac{1}{3}(a^2 + 2ah + h^2 + a^2 + ah + a^2)$

$= \frac{1}{3}(h^2 + 3ah + 3a^2)$

$= h\to 0 \quad a^2$

$-5x \to -5$

$\frac{\frac{-4a}{(a+h)} + \frac{4(a+h)}{a(a+h)}}{h} = \frac{-4a + 4a + 4h}{a(a+h)h} = \frac{4}{a^2+2ha+h^2}$

$= \frac{4}{a^2}$

$0 = x^2 - 5 + \frac{4}{x^2}$

$= x^4 - 5x^2 + 4$

$= (x^2-4)(x^2-1)$

$= (x-1)(x+1)(x-2)(x+2)$

$(2, \frac{-28}{3})$

$(-2, 28/3)$

$(1, -26/3)$

$(-1, 26/3)$