1.5 Properties of Limits

- For any real numbers a, suppose that f and g both have limits that exist at $x=a$

1) $\lim_{x\to a} k = k$ for any constant k

2) $\lim_{x\to a} x = a$

3) $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

4) $\lim_{x\to a} [cf(x)] = c[\lim_{x\to a} f(x)]$, for any constant c

5) $\lim_{x\to a} [f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$

6) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$

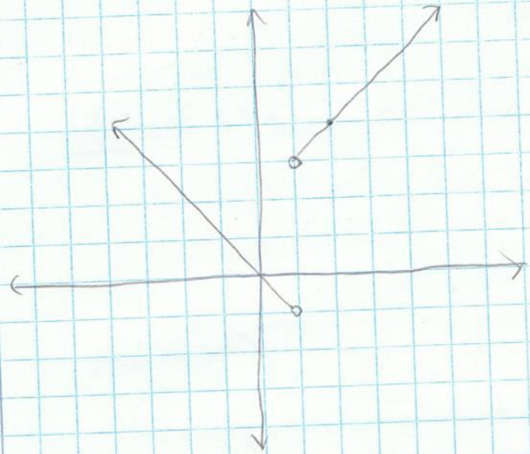
7) $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$, for any rational number n

- If f is a polynomial function, then $\lim_{x\to a} f(x) = f(a)$
- Substituting $x=a$ into $\lim_{x\to a} f(x)$ can yield the indeterminate form $\frac{0}{0}$.

If this happens, you may be able to find an equivalent function that is the same as the function f for all values except at $x=a$. Then, substitution can be used to find the limit

- To evaluate a limit algebraically, you can use the following techniques → direct substitution
  - → factoring
  - → rationalizing
  - → one-sided limits
  - → change of variable
- For any of these techniques, a graph or table of values can be used to check your result

17.



$$\frac{p^3\ (x) - 1}{\sqrt[3]{x} - 1}$$
p

$p = \sqrt[3]{x}$

$p^3 = x$

$$\lim_{x \to 1^+} \frac{(x^2 + x - 2)}{(x - 1)} = \lim_{x \to 1^+} \frac{(x + 2)\cancel{(x - 1)}}{\cancel{(x - 1)}}$$

$$\lim_{x \to 1} \frac{(x)\cancel{(x - 1)}}{-\cancel{(x + 1)}} = -x$$

1 To find the value of a limit:

A. Find $\lim_{x \to 1} \frac{(x-1)}{\sqrt{x} - 1}$ by rationalizing

B. Let $u = \sqrt{x}$ and rewrite $\lim_{x \to 1} \frac{(x-1)}{\sqrt{x} - 1}$ in terms of u.

We know $u^2 = x$, $\sqrt{x} \geq 0$. and $u \geq 0$. Therefore as x approaches the value of $\frac{1}{(x)}$, u approaches the value of 1. (x^2)

Use this substitution to find $\lim_{u \to 1} \frac{(u^2 - 1)}{u - 1}$ by reducing the rational expression.

1.5 Properties of Limits

$0.\overline{111}$ → convert to fraction

$x = 0.111$ $\qquad 10x - x = 1.111 - 0.111$

$10x = 1.1111$ $\qquad 9x = 1$

$x = \frac{1}{9}$

$x = 0.727272\ldots$

$100x = 72.7272\ldots$

$100x - \cancel{99}x = 72.7272 - 0.7272$

$99x = 72$

$x = \frac{72}{99}$

$0.123123123 = x$

$1000x = 123.123\ldots$

$1000x - x = 123$

$999x = 123$

$x = \frac{123}{999}$

ex. $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$

→ To find limit, find conjugate of numerator, x up and down

$$\lim_{x\to 7^+} \frac{\sqrt{x+2}-3}{x-7} \times \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} = \lim_{x\to 7} \frac{\cancel{x+2-9}}{\cancel{x-7}(\sqrt{x+2}+3)} = \lim_{x\to 7} \frac{1}{\sqrt{x+2}+3}$$

$$\lim_{x\to 7} \frac{1}{\sqrt{x+2}+3} = \frac{1}{\sqrt{7+2}+3} = \frac{1}{6}$$

$\therefore \lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} = \frac{1}{6}$ since $x \neq 7$, there is a hole at $(7, 1/6)$

→ can find limit sometimes without multiplying conjugate

This only works if answer $= \frac{0}{\text{value} \neq 0}$

ex. $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x+7}$

by substitution

$$\lim_{x\to 7} = \frac{\sqrt{7+2}-3}{x+7} = \frac{\sqrt{9}-3}{14} = \boxed{\frac{0}{14}}$$ $\therefore \lim_{x\to 7} = 0$

↳ this works, but if answer had been $\frac{0}{0}$, 0 would not have been the limit

ex. $\lim_{x\to 1} \frac{x^2-1}{x-1} = \boxed{\frac{0}{0}}$ ✗ → by substitution

• instead,

$$= \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = 1+1 = 2$$

hole at $(1, 2)$

· but plugging in $x=1$ into the beginning equation yields 0 as the answer

$\frac{(1)^2-1}{1-1}=\frac{0}{0}$ so how can the limit be a # other than 0?

PROOF →

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{0}{0}$$

$p(a)=0$, $q(a)=0$] Following the factor theorem, if $p(a)=0$, then $(x-a)$ is a factor of $p(x)$

$\therefore (x-a)$ is a factor of $q(x)$ too

$$\therefore \lim_{x\to a}\frac{\cancel{(x-a)}(\text{quotient})_1}{\cancel{(x-a)}(\text{quotient})_2}=\text{value}\neq 0$$

ex.3 $$\lim_{x\to 1}\frac{\sqrt{x}-1}{(x^2-1)}=\lim_{x\to 1}\frac{\sqrt{x}-1}{(x-1)(x+1)}\times\frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)(\sqrt{x}+1)}=\lim_{x\to 1}\frac{1}{(x+1)(\sqrt{x}+1)}=\frac{1}{(1+1)(\sqrt{1}+1)}=\frac{1}{4}$$

ex.4 $$\lim_{x\to -2}\frac{x^3+8}{x+2}$$

$$\lim_{x\to -2}\frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{x+2}}=\lim_{x\to -2}(x^2-2x+4)=(-2)^2-2(-2)+4$$
$$=4+4+4$$

$\therefore \lim_{x\to -2}=12$

$\therefore$ hole $(-2, 12)$ as well

Change of Variable technique

- also called substitution in textbooks
- used when all other forms of finding limit (sub, conjugate pairs, etc) don't work

ex. $$\lim_{x\to 0}\frac{(x+8)^{1/3}-2}{x}$$ $(x+8)^{1/3}$ is a problem, cannot expand b/c it's too long

- let $p=(x+8)^{1/3}$
- rewrite entire question using 'p' instead of x, even $x\to 0$ changes