4.3 Vertical + Horizontal Asymptotes

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→ vertical asymptote of a rational function $f(x) = \frac{p(x)}{q(x)}$

$q(c) = 0$, $p(c) \neq 0$, if there is a VA at $x = c$

- If $p(c)$ is also equal to zero, then at $x = c$, there is a hole

Ex. I $f(x) = \frac{1}{x^2 - 9}$ Find VA and behaviour around VA

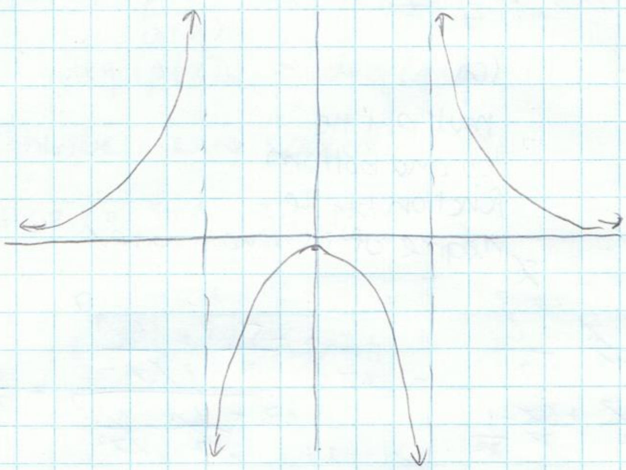
VA $= \pm 3$

$\lim_{x \to 3^+} f(x) = +\infty$ $\lim_{x \to 3^-} f(x) = -\infty$

$\lim_{x \to -3^+} f(x) = -\infty$ $\lim_{x \to -3^-} f(x) = +\infty$

+	+	−	−
$(-\infty,-3)$	$(-3,0)$	$(0,3)$	$(3,\infty)$

$(0, -1/9)$ max



- we confirm
  - max/min
  - end behaviours
  - intervals of inc/dec
  - behaviour around (VA)

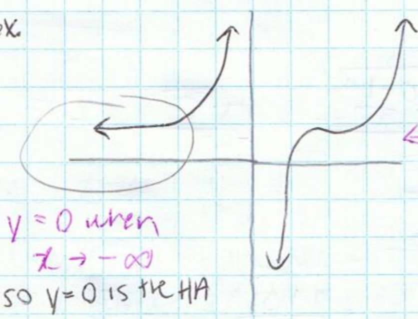
- asymptote does not cross x-axis b/c HA at $y=0$
- HA is $y=0$ when degree of denominator is higher than degree of numerator

Horizontal Asymptote: Behaviour of $f(x)$ when $x \to \pm\infty$

If $\lim\limits_{x\to\infty^+} f(x) = L$ <u>OR</u> $\lim\limits_{x\to\infty^-} f(x) = L$,

then $y = L$ is the HA.

ex.



$y=0$ when $x \to -\infty$
so $y=0$ is the HA

does not have to exist a $x \to \infty^+$

ex. $f(x) = \dfrac{2x+3}{x-1}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty} \frac{\frac{2x}{x} + \cancel{\frac{3}{x}}^{0}}{\frac{x}{x} - \cancel{\frac{1}{x}}^{0}} = \frac{2}{1} = 2$$

multiply the top and bottom function by the degree of the denominator x

$$\lim_{x\to\infty} \frac{\frac{-5x}{x^2} + \frac{6}{x^2}}{\frac{3x^2}{x^2} + \frac{6x}{x^2} - \frac{1}{x^2}} = \lim_{x\to\infty} \frac{\cancel{\frac{-5}{x}}^{0} + \cancel{\frac{6}{x^2}}^{0}}{3 + \cancel{\frac{6}{x}}^{0} - \cancel{\frac{1}{x^2}}^{0}} = \frac{0}{3} = 0$$

$\nearrow^{0}$ - means approaches zero

ex. $\displaystyle \lim_{x\to-\infty} \frac{\sqrt{5x^2+5x+1}}{x} = \lim_{x\to-\infty} \frac{\frac{5x^2}{x^2} + \cancel{\frac{5x}{x^2}}^{0} + \cancel{\frac{1}{x^2}}^{0}}{\frac{x^2}{x^2}} =$

$L^2 = 5$

$L = \pm\sqrt{5}$

$5 = \sqrt{25}$
$-5 = -\sqrt{25}$

$x\to\infty \quad x = \sqrt{x^2}$
$x\to-\infty \quad x = -\sqrt{x^2}$

$$\lim_{x\to-\infty} \frac{\sqrt{\frac{5x^2}{x^2} + \cancel{\frac{5x}{x^2}}^{0} + \cancel{\frac{1}{x^2}}^{0}}}{\frac{-\sqrt{x^2}}{x^2}} = \frac{\sqrt{5}}{-1} = -\sqrt{5}$$

$$= \frac{\frac{\sqrt{5x^2} + \cancel{5x}^{0} + \cancel{1}^{0}}{\sqrt{x^2}}}{\frac{-\sqrt{x^2}}{\sqrt{x^2}}} = \frac{\sqrt{5}}{-1} = -\sqrt{5}$$

Oblique Asymptote

- asymptote of the form $y = mx + b$

- $f(x) = \frac{p(x)}{q(x)}$ has an oblique asymptote if $\deg(p(x)) = \deg(q(x)) + 1$

- oblique means slanted

ex. 4 $f(x) = \frac{5x^2 + 8x - 7}{x - 1}$

$$\begin{array}{r|l} & 5x + 13 \\ \hline x - 1 & 5x^2 + 8x - 7 \\ & 5x^2 - 5x \\ \hline & \quad 13x - 7 \\ & \quad 13x - 13 \\ \hline & \qquad 6 \end{array}$$

oblique asymptote $\rightarrow y = 5x + 13$

- If there is an oblique asymptote, there is no HA

$f(x) = 5x + 13 + \frac{6}{x-1}$ $\qquad P(x) = q(x) + \frac{r(x)}{d(x)}$

$x \to 1^+ \quad y \to \infty$

1. interval of inc/dec $\rightarrow$ max + mins
2. y-intercept / x-intercepts
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4.3 Vertical + Horizontal Asymptotes Mar. 23. 2014

Vertical Asymptotes of Rational Functions

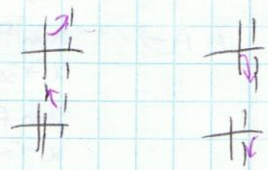
A rational function of the form $f(x) = \frac{p(x)}{q(x)}$ has a VA $x = c$ if $q(c) = 0$ and $p(c) \neq 0$.

Vertical Asymptotes and Infinite Limits

The graph of $f(x)$ has a vertical asymptote, $x = c$, if one of the following infinite limit statements is true:

$\lim_{x \to c^-} f(x) = +\infty$ $\lim_{x \to c^-} f(x) = -\infty$

$\lim_{x \to c^+} f(x) = +\infty$ $\lim_{x \to c^+} f(x) = -\infty$



The Reciprocal Function + Limits at Infinity

$\lim_{x \to +\infty} \frac{1}{x} = 0$ and $\lim_{x \to -\infty} 1/x = 0$

Horizontal Asymptotes and Limits at Infinity

If $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, we say that the line $y = L$ is a HA of the graph of $f(x)$.

- In a rational function, an **oblique asymptote** occurs when the degree of the numerator is exactly 1 greater than the degree of the denominator.

Algorithm for curve Sketching (so far)

1) Check discontinuities. Determine VAs and direction from which curves approach VA
2) Find both intercepts
3) Find any critical points
4) Use the first derivative test to determine the type of critical points that may be present
5) Test end behaviour by determining $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$
6) Construct an interval of $(+) \nearrow (-) \searrow$ table and identify all local or absolute extrema
7) Sketch the curve