4.4 · Concavity and Points of Inflection

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- The graph of a function $f(x)$ is **concave up** on an interval if $f'(x)$ is ↑ on the interval
- " " " **concave down** on an " " ↓ on the interval
- A point of inflection is a point on the graph of $f(x)$ where the function changes from concave up to concave down or up, or vice versa. $f''(c) = 0$ or is undefined if $(c, f(c))$ is a point of inflection on the graph of $f(x)$.

Test for Concavity

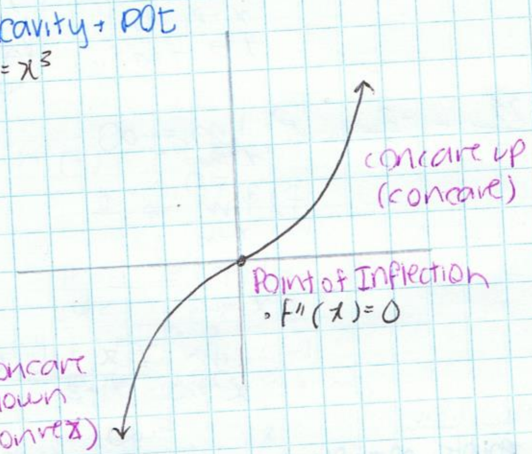
If $f(x)$ is a differentiable function whose second derivative exists on an open interval I, then
- the graph of $f(x)$ is concave up on I $f''(x)>0$ for all values of x in I
- the graph of $f(x)$ is concave down on I if $f''(x)<0$ for all values of x in I

The Second Derivative Test

Suppose that $f(x)$ is a function for which $f''(c)=0$, and the second derivative of $f(x)$ exists on an interval containing c.
- If $f''(c)>0$, then $f(c)$ is a loc. min.
- If $f''(c)<0$, then $f(c)$ is a loc. max
- If $f''(c)=0$, the test fails
  - use first derivative test

⑥ Concavity + POI

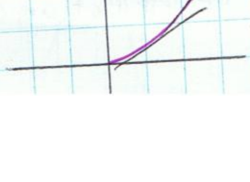
$y=x^3$



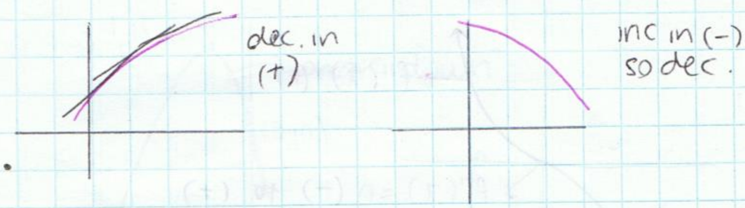
- slope dec.
- from horizontal tangent to vertical
- $f''(x)$ is $(+)$

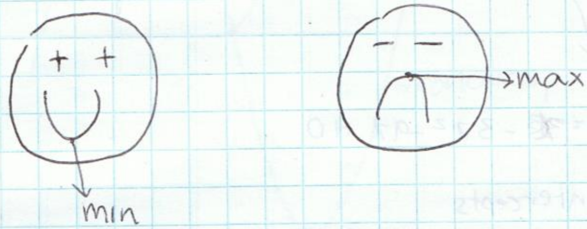
- slope inc.
- from vertical tangent to horizontal
- $f''(x)$ is $(-)$

- The graph of $y=f(x)$ is concave up on an interval $a\le x\le b$ in which the slopes ~~in which the slopes~~ of $f(x)$ are increasing



• The graph of $y = f(x)$ is concave down on an interval $a \le x \le b$ in which the slopes of $f(x)$ are decreasing





Second Derivative Test

• If $f''(x) > 0$ ~~min~~
$f''(x) < 0$ max

ex. $f(x) = x^3 - 6x^2 + 9x$

$f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x-1)(x-3)$

$x = 1, 3$ ← critical points

$f''(x) = 6x - 12$

$f''(1) = 6 - 12 < 0$

↳ max at $x = 1$

$f''(3) = 6(3) - 12 > 0$

$= 18 > 0$

↳ min at $x = 1$

second derivative to classify critical numbers

• If $x = c$ is a critical point ($f'(c) = 0$)

• If $f''(c) < 0 \Rightarrow x = c$ - there is a local max

$f''(c) > 0 \Rightarrow x = c$ there is a local min

Point of Inflection

• point on the graph of $y = f(x)$ when concavity changes

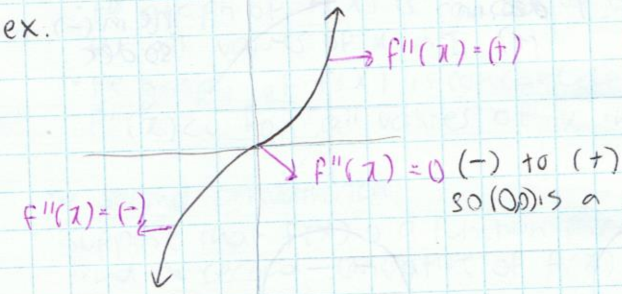
• must change signs before + after x-value when $y = f(x)$

b/c $f''(x) = 0$, x doesn't necessarily mean 'point of inflection'

• (concavity test) will check this.

- like when $f'(x)=0$, x-value isn't necessarily a max/min unless signs change to right + left.

ex.



ex. $f(x) = x^3 - 3x^2 - 9x + 10$

a) Find y-intercepts
b) Find intervals of inc/dec
c) Classify the critical points
d) Find intervals of concave up/down
e) Find POI
f) Sketch the graph

a) $f(0) = 10$
$(0, 10)$

b) $f'(x) = 3x^2 - 6x - 9$
$= 3(x^2 - 2x - 3)$
$3(x-3)(x+1)$
$x = +3, -1$
$(-1, 15)$
$(3, -17)$

	$-\infty, -1$	$-1, 3$	$3, \infty$
	+	−	+

inc → $x < -1$, $x > 3$
dec → $-1 < x < 3$

c) $f''(x) = 6x - 6$
$f''(+3) > 0$
Lo min

$f''(-1) = 6 - 6$
$f''(-1) < 0$
Lo max

d) $0 = 6x - 6$
$x = 1$

$(1, -1)$

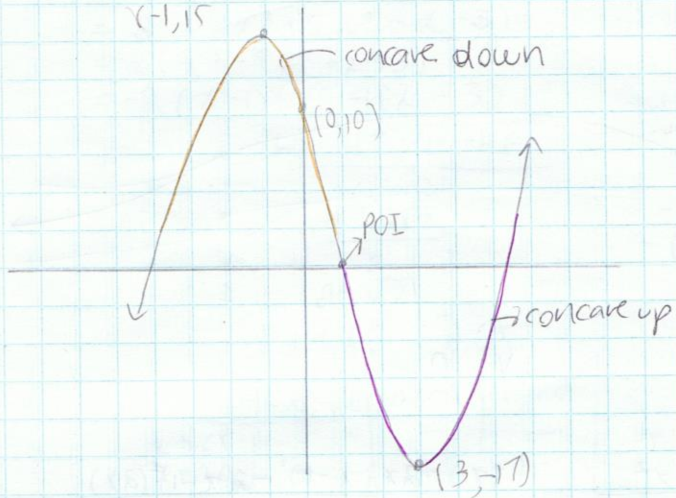
	$-\infty, 1$	$1, \infty$
	−	+

concave down $x < 1$
" up $x > 1$

e) $(-) \rightarrow (+)$

so $(1, -1)$ is a POI

f)



2. ex. 2 $\quad f(x) = \dfrac{1}{x^2+3}$

3 ex. 3 $\quad f(x) = \dfrac{x^2}{(x-1)^2}$

2. a) $0 = \dfrac{1}{x^2+3}$ $\qquad y = \dfrac{1}{3}$

no x intercepts

b) $f'(x) = \dfrac{d}{dx}(x^2+3)^{-1}$

$= -1(x^2+3)^{-2}(2x)$

$0 = \dfrac{-2x}{(x^2+3)^2}$

$x = 0$

$-\infty, 0$	$0, \infty$
+	−

$\rightarrow 0 \rightarrow$ max

inc; $x \in \mathbb{R} \mid x \leq 0$

dec; $x \in \mathbb{R} \mid x > 0$

c) $f''(x) = -2(x^2+3)^{-2} + (-2)(x^2+3)^{-3}(2x)(-2x)$

$0 = (x^2+3)^{-3}(-2(x^2+3) + 8x^2)$

$= (x^2+3)^{-3}(-2x^2 - 6 + 8x^2)$

$0 = (x^2+3)^{-3}(6x^2 - 6)$

$\Downarrow$

$x = \pm 1$

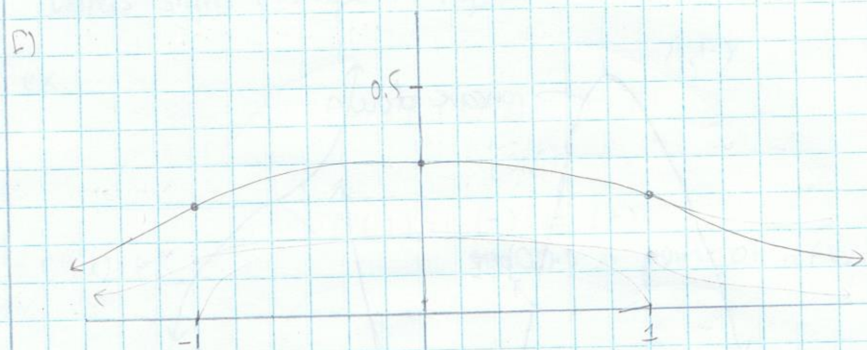
$-\infty, -1$	$-1, 1$	$1, \infty$
+	−	+

concave up: $x \in \mathbb{R}; x < -1$, $x > 1$

concave down: $x \in \mathbb{R} \mid -1 < x < 1$

$x = -1, 1$

f)



3. a) $0 = x^2$ $\quad y = 0$

$x = 0$

b) $f(x) = \dfrac{x^2}{(x-1)^2}$ $\quad f'(x) = \dfrac{2x(x-1)^2 - 2(x-1)^4(2x)}{(x-1)^4}$

$0 = \dfrac{(x-1)[2x(x-1) - 4x]}{(x-1)^4}$

	$-\infty, 0$	$0, 1$	$1, 3$	$3, \infty$
$2x$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x-3$	$-$	$-$	$-$	$+$
	$-$	$+$	$-$	$+$

$x = 1$ $\quad 2x^2 - 2x - 4x$

$x = 0$ $\quad 2x^2 - 6x$

$x = 3$ $\quad 2x(x-3)$

(1, undefined)
(0, 0)
(3, 9/4)

inc $\to$ $x > 3$ $\quad 0 < x < 1$

dec $\to$ $x < 0$ $\quad 1 < x < 3$

c) $0 \to$ min

$1 \to$ asymptote

$3 \to$ neither

d) $f''(x) = \dfrac{(x-1)(2x^2 - 6x)}{(x-1)^4}$

$= \dfrac{2x^3 - 2x^2 - 6x^2 + 6x}{(x-1)^4}$

$= \dfrac{2x^3 - 8x^2 + 6x}{(x-1)^4}$

$0 = \dfrac{(6x^2 - 16x + 6)(x-1)^4 - (4)(x-1)^3(2x^3 - 8x^2 + 6x)}{(x-1)^8}$

~~$3x^2 - 9x + x + 3$~~
~~$3x(x$~~
$\dfrac{8 \pm \sqrt{8^2 - 4(3)(3)}}{6}$
2.2

$= \dfrac{2(3x^2 - 8x + 3)(x-1)^4 - 4(x-1)^3(\cancel{2x})(x^2 - 4x + 3)}{(x-1)^8}$

d) $0 = 2\cancel{(x-1)^3}\left[(3x^2-8x+3)(x-1) - 4x(x^2-4x+3)\right]$

$= 2(\cancel{3x^3} - \cancel{8x^2} + \cancel{3x} - \cancel{3x^2} + \cancel{8x} - 3 \cancel{-4x^3} + \cancel{16x^2} \cancel{-12x})$

$= 2(-x^3 + 5x^2 - x - 3)$

$= -2(x^3 - 5x^2 + x + 3)$

$= -2(x-1)(x^2 - 4x - 3)$

$\hookrightarrow \dfrac{4 \pm \sqrt{16 - 4(-3)}}{2}$

$\dfrac{4 \pm 2\sqrt{7}}{2}$

$2 \pm \sqrt{7}$

$1,\ 4.65,\ -0.65$

1	-1	5	-1	-3
	↓	-1	4	3
	-1	4	3	0

$-x^2 + 4x + 3$

$-(x^2 - 4x - 3)$

	$-\infty, -0.65$	$-0.65, 1$	$1, 4.65$	$4.65, \infty$
$x-4.65$	$-$	$-$	$-$	$+$
$x+0.65$	$-$	$+$	$+$	$+$
$2(x-1)$	$-$	$-$	$+$	$+$
	$-$	$+$	$-$	$+$

concave up $\to$ ~~x~~ $x < 1$, $x > 4.65$

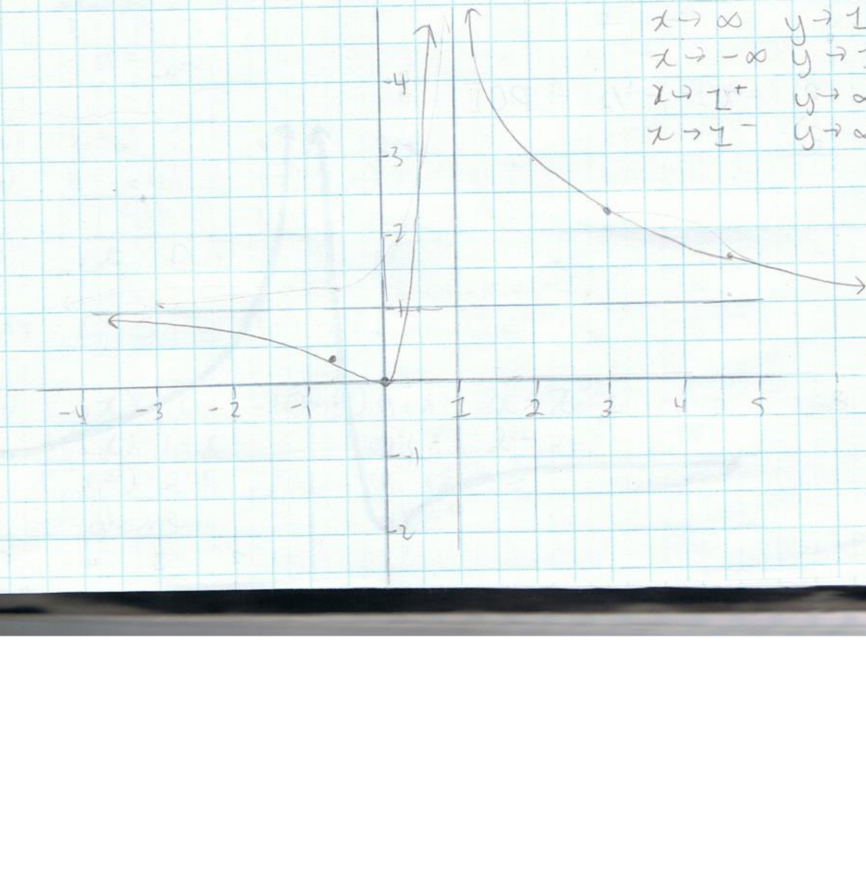
concave down $\to$ $1 < x < 4.65$, $x < -0.65$

e) $1 \to VA$ $(1, \text{undefined})$

$-0.65 \to POI$ $(-0.65, 0.16)$

$4.65 \to POI$ $(4.65, 1.62)$

f)



$x \to \infty$ $y \to 1^+$

$x \to -\infty$ $y \to 1^-$

$x \to 1^+$ $y \to \infty$

$x \to 1^-$ $y \to \infty$

$$f'(x) = \frac{2x(x-1) \quad -2x^2}{(x-1)^3}$$

$$0 = \frac{\cancel{2x^2} - 1 - \cancel{2x^2}}{(x-1)^3}$$

$$= -1$$

no min or max

$x = 0$
$x = 1$

$-\infty, 0$	$0, 1$	$1, \infty$
$-$	$+$	$-$

inc $\rightarrow 0 < x < 1$
dec $\rightarrow x < 0$, $x > 1$

c) $x = 0 \leftarrow$ min $(0, 0)$
$x = 1 \leftarrow$ asymptote $(1,$ undefind$)$

d) $$f''(x) = \frac{6x^2 - 8x(x-1) + 2(x-1)^2}{(x-1)^4}$$

$$0 = 6x^2 - 8x^2 + 8x + 2(x^2 - 2x + 1)$$
$$= \cancel{6x^2} - \cancel{8x^2} + 8x + \cancel{2x^2} - 4x + 2$$
$$= 4x + 2$$
$$x = -1/2$$

$(-1/2, 1/9)$

$-\infty, -1/2$	$-1/2, \infty$
$-$	$+$

e) $x = -1/2 \rightarrow$ POI

f)